Analytic number theory
Solutions to Exercise Sheet 4

Exercise 1. Note that if $p|n$ and $p > \sqrt{n}$, then p is the unique prime with this property, i.e. if $q|n$, is a prime dividing n, then either $q = p$ or $q \leq \sqrt{n}$, in fact suppose $q \neq p$ and $q > \sqrt{n}$, then $qp|n$ and at the same time $qp > n$ which is a contradiction. For the same reason $p^2 \nmid n$. On the other hand for each prime $p \leq x$ and each integer $a < p$ the number pa has a prime divisor bigger than its square root (namely p). In particular we get

$$S(x) = \sum_{n \leq x} 1 - \sum_{p \leq x} \sum_{\substack{a < p \\ ap \leq x}} 1.$$

We focus on the second summation: notice that if $p \leq x^{1/2}$, then the condition $a < p$ implies the condition $ap \leq x$. If otherwise $p > x^{1/2}$ then the condition $ap \leq x$ implies the condition $a < p$. In particular we can write

$$\sum_{p \leq x} \sum_{\substack{a < p \\ ap \leq x}} 1 = \sum_{p \leq x^{1/2}} \sum_{a < p} 1 + \sum_{x^{1/2} < p \leq x} \sum_{a \leq x/p} 1. \tag{0.1}$$

We have esimating crudely (in the first step)

$$\sum_{p \leq x^{1/2}} \sum_{a < p} 1 \leq \sqrt{x} \sum_{p \leq x^{1/2}} 1 = O(x/\log x),$$

the latter following by Chebyschev's Theorem applied to $\pi(\sqrt{x})$. For the second summand on the right hand side of (0.1) we get:

$$\begin{aligned}
\sum_{x^{1/2} < p \leq x} \sum_{a \leq x/p} 1 &= \sum_{x^{1/2} < p \leq x} \lfloor x/p \rfloor \\
&= x \sum_{x^{1/2} < p \leq x} \frac{1}{p} + \sum_{x^{1/2} < p \leq x} O(1) \\
&\overset{\dagger}{=} x(\log\log x + C - \log\log x^{1/2} - C + O(1/\log x)) + O(x/\log x) \\
&= (\log 2)x + O(x/\log x),
\end{aligned}$$

using that $\log\log x^{1/2} = \log\log x - \log 2$. In † we apply the stated version of Merten's Theorem for the first summand and the Chebyschev's Theorem for the second.

We conclude:

$$S(x) = \sum_{n \leq x} 1 - (\log 2)x + O(x/\log x) = x(1 - \log 2) + O(x/\log x).$$

Exercise 2. We have

$$\begin{aligned}\sum_{n\le x}\omega(n) &= \sum_{p\le x}\sum_{\substack{n\le x\\ p|n}} 1\\ &= \sum_{p\le x}\sum_{a\le x/p} 1\\ &= \sum_{p\le x}\lfloor x/p\rfloor\\ &= x\sum_{p\le x}\frac{1}{p}+\sum_{p\le x}O(1)\\ &= x\log\log x + Cx + O(x/\log x),\end{aligned}$$

using both Merten's and Chebyschev's Theorem.

For $\Omega(n)$ we compute similarly

$$\begin{aligned}\sum_{n\le x}\Omega(n) &= \sum_{p\le x}\sum_{\substack{n\le x\\ p|n}}\sum_{0<l:p^l|n} 1\\ &= \sum_{p\le x}\sum_{1\le l\le \frac{\log x}{\log p}}\lfloor x/p^l\rfloor\\ &= \sum_{p\le x}\lfloor x/p\rfloor + \sum_{p\le x^{1/2}}\sum_{2\le l\le \frac{\log x}{\log p}}\lfloor x/p^l\rfloor,\end{aligned}$$

where for the second summation we used that if p contributes to the sum, then $p < x^{1/2}$. The first summation is $\sum_{n\le x}\omega(n)$. We proceed in analyzing the second further:

$$\begin{aligned}\sum_{p\le x^{1/2}}\sum_{2\le l\le \frac{\log x}{\log p}}\left(\frac{x}{p^l}+O(1)\right) &= x\sum_{p\le x^{1/2}}\sum_{2\le l\le \frac{\log x}{\log p}}\frac{1}{p^l}+O(\log x\sum_{p\le x^{1/2}} 1)\\ &\overset{\dagger}{=} x\sum_{p\le x^{1/2}}\sum_{2\le l\le \frac{\log x}{\log p}}\frac{1}{p^l}+O(x^{1/2}).\end{aligned}$$

Notice that

$$\begin{aligned}\sum_{2\le l\le \frac{\log x}{\log p}}\frac{1}{p^l} &= \sum_{2\le l\le\infty}\frac{1}{p^l}-\sum_{\frac{\log x}{\log p}<l<\infty}\frac{1}{p^l}\\ &= \frac{1}{1-1/p}-1-1/p-\frac{p^{-\frac{\log x}{\log p}-1}}{1-1/p}\\ &= \frac{p^2-p(p-1)-(p-1)}{p(p-1)}-x^{-1}\frac{1}{p-1}\\ &= \frac{1}{p(p-1)}-x^{-1}\frac{1}{p-1}.\end{aligned}$$

We obtain therefore

$$
\begin{aligned}
\sum_{p\le x^{1/2}}\sum_{2\le l\le \frac{\log x}{\log p}}\left(\frac{x}{p^l}+O(1)\right) &= \sum_{p\le x^{1/2}}\left(x\frac{1}{p(p-1)}-\frac{1}{p-1}\right)+O(x^{1/2})\\
&= x\sum_{p\le x^{1/2}}\frac{1}{p(p-1)}+O(x^{1/2})\\
&= x\sum_{1<p<\infty}\frac{1}{p(p-1)}-x\sum_{p>x^{1/2}}\frac{1}{p(p-1)}+O(x^{1/2}).
\end{aligned}
$$

Now notice that $\sum_{p>x^{1/2}}\frac{1}{p(p-1)}=O(x^{-1/2})$ and denote $c=\sum_{1<p<\infty}\frac{1}{p(p-1)}<\infty$. Summarizing everything we have

$$\sum_{n\le x}\Omega(n)=\sum_{n\le x}\omega(n)+cx+O(x^{1/2}).$$

Exercise 3. By definition

$$\Theta(x)=\sum_{p\le x}\log p=\log\left(\prod_{p\le x}p\right)=\log\left(C(x)\right)\implies C(x)=e^{\Theta(x)}.$$

Let $p_1,...,p_k$ be all the primes $\le x$ and $p_i^{r_i}\le x<p_i^{r_i+1}$. Then by definition of $\Lambda(n)$,

$$
\begin{aligned}
\psi(x) &= \sum_{n\le x}\Lambda(n)\\
&= \sum_{p_i\in\{p_1,\dots,p_k\}}\sum_{1\le n\le r_i}\Lambda(p_i^n)\\
&= \sum_{p_i\in\{p_1,\dots,p_k\}}\sum_{1\le n\le r_i}\log(p_i)\\
&= \log(\prod_{1\le i\le k}p_i^{r_i})\\
&= \log D(x)\\
&\implies D(x)=e^{\Psi(x)}.
\end{aligned}
$$

Exercise 4. (a)

$$
\begin{aligned}
\Phi(C(y),y) &= \#\{n\le C(y)\ :\ p|n\implies p>y\}\\
&= \#\{n\le C(y)\ :\ p\le y\implies p\nmid n\}\\
&= \#\{n\le C(y)\ :\ (n,C(y))=1\}\\
&= \phi(C(y))=C(y)\prod_{p<y}(1-p^{-1})=C(y)\left(\frac{A}{\log y}+\mathrm{O}\left(\frac{1}{(\log y)^2}\right)\right),
\end{aligned}
$$

where for the last line we have used what has been done in the class. This gives the desired result.

Remark 1. Or we can do in other way that we were discussing in the class. For that we let $A_d = \{n \leq C(y) \mid d|n\}$. Then we have by inclusion exclusion principle,

$$\begin{aligned} C(y) - \Phi(C(y), y) &= |\cup_{p\leq y} A_p| \\ &= -\sum_{d|C(y)} \mu(d)|A_d| + |A_1| \\ &= -\sum_{d|C(y)} \mu(d)\lfloor\frac{C(y)}{d}\rfloor + C(y) \\ &\implies \Phi(C(y), y) = \sum_{d|C(y)} \mu(d)\frac{C(y)}{d} + \mathrm{O}(\sum_{d|C(y)} 1) \\ &\implies \Phi(C(y), y) = C(y)\prod_{p\leq y}(1 - p^{-1}) + \mathrm{O}(2^{\Pi(y)}) \\ &\implies \Phi(C(y), y) = C(y)\prod_{p\leq y}(1 - p^{-1}) + \mathrm{O}(e^{y/\log y}). \end{aligned}$$

Here one can check by using the derivative and try to understand that that $e^{y/\log y} \ll e^y/(\log y)^2 \sim C(y)/(\log y)^2$.

(b) From the previous exercise, $C(y) = e^{\Theta(y)} \sim e^y$, from Chebyshev's theorem, done in class. But for our case, there exists sufficiently large C such that $x \geq e^{Cy} \implies x \sim e^{Cy} \sim C(y)^C$. But for us $\Phi(C(y), y) \sim \frac{C(y)A}{\log y} \implies \Phi(C(y), y) \sim \frac{C(y)^C A}{\log y} \implies \Phi(x, y) \sim \frac{xA}{\log y}$.

Exercise 5. $(a) \Rightarrow (b)$: By assumption there exists $C > 0$ such that for all $n \geq 1$ we have

$$|f(n)| \leq Cn^A.$$

Let $\mathrm{Re}(s) > A + 1$ and $\delta = \mathrm{Re}(s) - A - 1 > 0$

$$|\sum_{n=1}^{\infty} \frac{f(n)}{n^s}| \leq C\sum_{n=1}^{\infty} n^{A-\mathrm{Re}(s)} \leq \sum_{n=1}^{\infty} n^{-(1+\delta)} < \infty.$$

In particular $\sigma_a(f) \geq A + 1$.

$(b) \Rightarrow (a)$: Suppose $\sigma_a(f) < \infty$. In particular for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \sigma_a(f)$ we have that

$$\sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\mathrm{Re}(s)}} < \infty.$$

Fix any s with $\mathrm{Re}(s) > \sigma_a(f)$. The absolute convergence implies that $\lim_{n\to\infty} \frac{|f(n)|}{|n^s|} = 0$. In particular there exists a $N_0 \in \mathbb{N}$ such that for all $n > N_0$ it holds that $|f(n)| \leq n^{\mathrm{Re}(s)}$. Let $M = \max_{n=1,\ldots,N_0} \frac{|f(n)|}{n^{\mathrm{Re}(s)}}$. By setting $C = \max(M, 1)$ we get that for all $n \geq 1$:

$$|f(n)| \leq Cn^{\mathrm{Re}(s)}$$

and we may want to choose $A = \mathrm{Re}(s)$.